Equivalence of the Potts model or Whitney polynomial with an ice-type model

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Abstract. The partition function of the Potts model on any lattice can readily be written as a Whitney polynomial. Temperley and Lieb have used operator methods to show that, for a square lattice, this problem is in turn equivalent to a staggered ice-type model. Here we rederive this equivalence by a graphical method, which we believe to be simpler, and which applies to any planar lattice. For instance, we also show that the Potts model on the triangular or honeycomb lattice is equivalent to an ice-type model on a Kagomé lattice.

1. Introduction

There is at present considerable interest amongst statistical mechanics and combinatorialists in the evaluation of the Whitney polynomial of a graph. This is because this problem is the same as obtaining the partition function of the Potts (1952) model on the graph (Kasteleyn and Fortuin 1969, Fortuin and Kasteleyn 1972, Baxter 1973), while in addition it contains the percolation and colouring problem as special cases.

Some exact results are available for the square lattice graph. In particular, when the associated Potts model has two states per spin, it becomes the Ising model and the problem is soluble. Also, Temperley and Lieb (1971) have established a remarkable equivalence between the Whitney polynomial for a square lattice $\mathscr{L}$ and the partition function of a staggered ice-type model on a related square lattice $\mathscr{L}'$.

Although the polynomial has not yet been evaluated exactly for the square lattice, it is tempting to think that it may be. The problem has therefore attracted attention amongst theoreticians, to the extent that we feel it worthwhile presenting a rederivation of the equivalence established by Temperley and Lieb. We use graphical methods which we believe to be simpler than the operator method of Temperley and Lieb. Further, they apply to any planar lattice, regular or not.

2. Potts model and Whitney polynomial

First we define the q-state Potts model. There is more than one model by this name (Potts 1952, Domb 1974); the one we use here is the 'scalar' rather than the 'vector' model.

Let $\mathscr{L}$ be any lattice and with each site i associate a spin σ_i with values $1, 2, \ldots, q$. Let nearest-neighbour spins have interaction energy $-\epsilon$ if they are alike, zero if they are different. Then the partition function is

$$Z = \sum \exp\left(\beta\epsilon \sum_{(i,j)} \delta(\sigma_i, \sigma_j) \right) \tag{1}$$

where the summation inside the exponential is over all nearest-neighbour pairs (i,j) on the lattice. The summation outside is over all states of the spins. If there are N sites, then there are q^N spin states of the system.

Set

$$v = e^{\beta\epsilon} - 1, \tag{2}$$

then

$$Z = \sum \prod_{(i,j)} (1 + v\delta(\sigma_i, \sigma_j)). \tag{3}$$

Let E be the number of edges of the lattice. Then the summand in (3) is a product of E factors, and expanding the product gives 2^E terms.

To each term we associate a bond-graph in $\mathscr{L}$ by placing bonds on edges where we have taken the corresponding $v\delta(\sigma_i, \sigma_j)$ term in the expansion. If we take the unit term, we leave the corresponding edge empty.

This gives a one-to-one correspondence between terms in the expansion of the summand of (3), and graphs on $\mathscr{L}$.

Consider a typical graph G, containing l bonds and C connected components (regarding an isolated site as a component). Then the corresponding term in (3) contains a factor v^l, and the effect of the delta functions is that all sites within a component must have the same spin σ. Summing over all independent spins therefore gives

$$Z = \sum_G q^C v^l, \tag{4}$$

where the summation is over all the 2^E graphs G that can be drawn on $\mathscr{L}$. The expression (4) is a Whitney (1932) polynomial.

It is easy to see that (4) contains the percolation and colouring problems as special cases. In particular,

$$\left(\frac{\partial}{\partial q} \ln Z \right)_{q=1}$$

is the mean number of components of the percolation problem. Also, if $\epsilon = -\infty$ and $v = -1$, then the spins (or colours) of adjacent sites must be different, and Z becomes the q-colouring polynomial of the lattice.

The edges of regular lattices can be grouped naturally into certain classes. For instance the square lattice has edges which are either horizontal or vertical. It is then natural and convenient to generalize (1)–(4) so as to allow different values of the interaction energy $-\epsilon$, according to which class the corresponding edge belongs. If ϵ_r is the value of ϵ for edges of class r, and

$$v_r = \exp(\beta\epsilon_r) - 1, \tag{5}$$

then the required generalization of (4) is easily seen to be:

$$Z = \sum_G q^C v_1^{l_1} v_2^{l_2} v_3^{l_3} \ldots \qquad (6)$$

where the summation is over all graphs G, C is the number of connected components in G, and l_r is the number of bonds on edges of class r ($r = 1, 2, 3, \ldots$).

3. Planar lattices: the surrounding lattice $\mathscr{L}'$

The remarks of § 2 apply to any lattice $\mathscr{L}$, whatever its structure or dimensionality. From now on we specialize to $\mathscr{L}$ being a planar lattice. It does not have to be regular, but can be any finite set of points (sites) and straight edges linking pairs of points. Points which are linked by an edge are said to be 'neighbours' or 'adjacent'. Planar means that no two edges cross.

We associate with $\mathscr{L}$ another planar lattice $\mathscr{L}'$, as follows.

Draw simple polygons surrounding each site of $\mathscr{L}$ such that:

(i) no polygons overlap, and no polygon surrounds another;

(ii) polygons of non-adjacent sites have no common corner;

(iii) polygons of adjacent sites i and j have one and only one common corner. This corner lies on the edge (i, j).

We take the corners of these polygons to be the sites of $\mathscr{L}'$, and the edges to be the edges of $\mathscr{L}'$. Hereinafter we call these polygons the 'basic polygons' of $\mathscr{L}'$.

We see that there are two types of sites of $\mathscr{L}'$. Firstly, those common to two basic polygons. These lie on edges of $\mathscr{L}$ and have four neighbours in $\mathscr{L}'$. We term these 'internal' sites. Secondly, there can be sites lying on only one basic polygon. These have two neighbours and we term them 'external' sites. (The reason for this terminology will become apparent when we explicitly consider the regular lattices.)

The above rules do not determine $\mathscr{L}'$ uniquely, in that its shape can be altered, and external sites can be added on any edge. However, the topology of the linkages between internal sites is invariant, and the general argument of the following sections applies to any allowed choice of $\mathscr{L}'$. (For the regular lattices there is an obvious natural choice.) In figure 1 we show an irregular lattice $\mathscr{L}$ and its surrounding graph $\mathscr{L}'$.
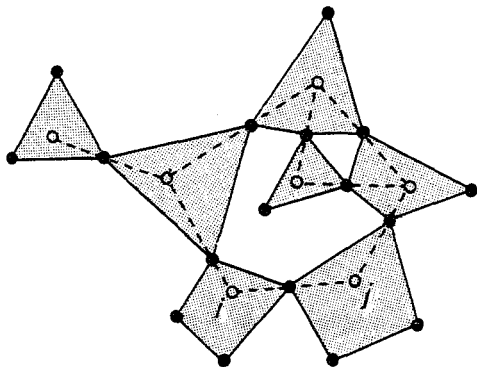

Figure 1. An irregular lattice $\mathscr{L}$ (open circles and broken lines) and its surrounding lattice $\mathscr{L}'$ (full circles and lines). The interior of each basic polygon is shaded, denoting 'land'.

It is helpful to shade the interior of each basic polygon, as in figure 1, and to regard such shaded areas as 'land', unshaded areas as 'water'. Then $\mathcal{L}'$ consists of a number of 'islands'. Each island contains a site of $\mathcal{L}$. Islands touch on edges of $\mathcal{L}$, at internal sites of $\mathcal{L}'$.
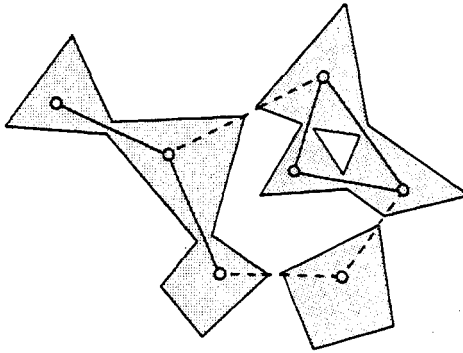
4. Polygon decompositions of $\mathcal{L}'$

We now make a one-to-one correspondence between graphs G on $\mathcal{L}$ and decompositions of $\mathcal{L}'$ as follows.

If G does not contain a bond on an edge (i, j), then at the corresponding internal site of $\mathcal{L}'$ separate two edges from the other two so as to separate the islands i and j, as in figure 2(a). If G contains a bond, separate the edges so as to join the islands, as in figure 2(b). Do this for all edges of $\mathcal{L}$.
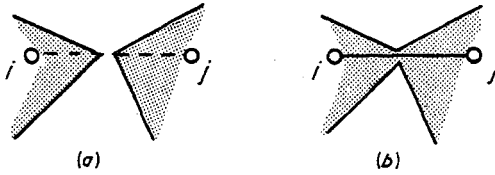
Figure 2. The two possible separations of the edges at an internal site of $\mathcal{L}$ (lying on the edge (i, j) of $\mathcal{L}$). The first represents no bond between i and j, the second a bond.

The effect of this is to decompose $\mathcal{L}'$ into a set of disjoint polygons, an example being given in figure 3. (We now use 'polygon' to mean any simple closed polygonal path on $\mathcal{L}'$.)

Figure 3. A graph G on $\mathcal{L}$ (full lines between open circles represent bonds), and the corresponding polygon decomposition of $\mathcal{L}'$. To avoid confusion at internal sites, sites of $\mathcal{L}$ are not explicitly indicated, but are to be taken to be in the same positions as in figure 1.

Clearly any connected component of G now corresponds to a large island in $\mathcal{L}'$, made up of basic islands joined together. Each such large island will have an outer perimeter, which is one of the polygons into which $\mathcal{L}'$ is decomposed. A large island may also contain lakes within; these correspond to circuits of G and also have a polygon as outer perimeter.

Each polygon is of one of these two types. Thus $\mathcal{L}'$ is broken into p polygons, where

$$p = C + S, \tag{7}$$

and C and S are, respectively, the number of connected components and circuits in G. If $\mathcal{L}$ has N sites, then Euler's relation gives

$$S = C - N + l_1 + l_2 + l_3 + \ldots. \tag{8}$$

Eliminating S and C from the above equations (6), (7), (8), it follows that

$$Z = q^{N/2} \sum q^{p/2} x_1^{l_1} x_2^{l_2} x_3^{l_3} \ldots, \tag{9}$$

where

$$x_r = q^{-1/2} v_r \tag{10}$$

and we now take the summation to be over all polygon decompositions of $\mathcal{L}'$. Here p is the number of polygons in the decomposition, and l_r is the number of internal sites of class r where the edges have been separated as in figure 2(b).

5. Equivalent ice-type model on $\mathcal{L}'$

In this section we first define an ice-type model (Lieb 1967) on the lattice $\mathcal{L}'$, and state that its partition function is $q^{-N/2} Z$. We then prove this equivalence.

Let θ and z be two parameters given by

$$q^{1/2} = 2 \cosh \theta, \qquad z = \exp(\theta/2\pi). \tag{11}$$

Then the ice-type model is defined as follows.

(a) Place arrows on the edges of $\mathcal{L}'$ so that at each site an equal number of arrows point in and out.

(b) With each external site associate a weight z^α if an observer moving in the direction of the arrows turns through an angle α to his left, or an angle $-\alpha$ to his right, as he goes through the site. This angle α is shown in figure 4.
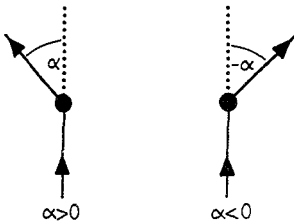

Figure 4. External sites of $\mathcal{L}'$ at which an observer moving in the direction of the arrows turns through an angle α to his left, or equivalently an angle $-\alpha$ to his right. Note that $-\pi < \alpha < \pi$, and the angle between the edges is $\pi - |\alpha|$.

(c) There are six possible arrangements of arrows at an internal site, as shown in figure 5. With arrangement k on a site of class r associate a weight ω_k, where

$$\omega_1, \ldots, \omega_6 = z^{\alpha-\gamma}, \; z^{\gamma-\alpha}, \; x_r z^{\beta-\delta}, \; x_r z^{\delta-\beta}, \; z^{-\beta-\delta} + x_r z^{\alpha+\gamma}, \; z^{\beta+\delta} + x_r z^{-\alpha-\gamma} \tag{12}$$

and the angles α, β, γ, δ are those shown in figure 5.
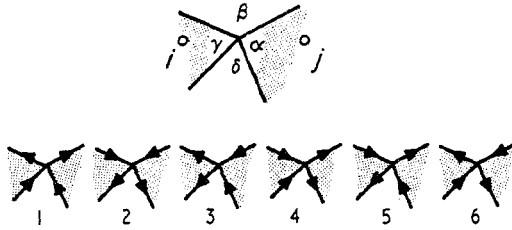
Figure 5. A typical internal site of $\mathscr{L}'$, showing the angles between edges and the six possible arrangements of arrows. When using (12) and this figure it is important to note that the angles α and γ lie inside basic polygons ('islands') of $\mathscr{L}'$, while β and δ lie outside.

The partition function of this ice-type model is

$$Z' = \sum \prod (\text{weights}),\tag{13}$$

where the product is over all sites of $\mathscr{L}'$ and the summation over all allowed arrow configurations on $\mathscr{L}'$. We shall prove that

$$Z = q^{N/2} Z',\tag{14}$$

where Z is the Potts model partition function (or Whitney polynomial) defined and discussed in the preceding sections.

5.1. Proof of equivalence

Take a polygon decomposition of $\mathscr{L}'$ and place arrows on the edges so that at each corner there is one pointing in and one pointing out. Give a polygon corner a weight z^{α}, where α is the angle to the left through which an observer moving in the direction of the arrows turns when passing through the corner. Since edges cannot overlap, α must lie in the interval $-\pi < \alpha < \pi$.

Since each polygon is a simple closed curve, on moving right round the polygon this observer turns through a total angle $\pm 2\pi$, depending on whether the arrows point anticlockwise or clockwise. Both cases occur, so this rule gives a polygon a total weight

$$z^{2\pi} + z^{-2\pi}.\tag{15}$$

The conditions (11) ensure that this is $q^{1/2}$, which from (9) is precisely the weight we wish to associate with each polygon. It follows from (9) that $q^{-N/2} Z$ can be obtained by the following procedure.

(A) At each internal site of $\mathscr{L}'$ separate the edges, either as in figure 2(a) or (b). If the latter, associate a weight x_r, where r is the class of the site.

(B) Place arrows on the edges round each site (internal and external), so as to follow one another round the polygon corners. Associate the appropriate weight z^{α} with each corner.

(C) Do (A) and (B) *independently for each site*. Then require that on an edge (i, j) there cannot be an arrow pointing into (or out of) both sites i and j. Reject all configurations that fail this requirement on any edge.

(D) Sum over all remaining configurations thus obtained, weighted by the product of the individual weights.

Since (A) and (B) can be performed independently for each site, they can be combined into:

(AB) Place arrows on the edges surrounding each site in any configuration that is generated by rules (A) and (B). Give each such configuration a weight equal to the product of the x_r and corner weights given by rules (A) and (B), *summed over all ways (A) and (B) give this arrow configuration.*

Since (A) and (B) can only give configurations with an equal number of arrows into and out of each site, the new rules (AB), (C), (D) define an ice-type model on $\mathscr{L}'$, with site weights given by (AB).

At external sites rule (B) immediately gives the rule (b) that we wish to establish. At internal sites (A) and (B) give eight possibilities, as shown in figure 6, but we note that the last two resulting arrow configurations are the same in the top and bottom rows. Calculating the weights from rules (A) and (B), the resulting total weights for the six distinct configurations are those given by equation (12) of rule (c). Thus the ice-type model defined by (AB), (C), (D) is the same as that given by (a), (b), (c). This proves the equation (14) and establishes the desired equivalence.
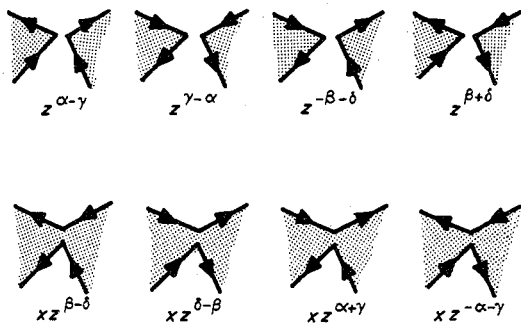

Figure 6. The two possible separations of edges at an internal site of $\mathscr{L}'$, and the eight allowed arrangements of arrows thereon. The product of the weights given by rules (A) and (B) is shown underneath, using the notation of figure 5 and omitting the suffix of x_r.

5.2. Four-colouring problem

It is fascinating to wonder whether this equivalence brings us any closer to a proof of the famous four-colour conjecture. From (11), $q = 4$ is a 'critical' case, since for $q < 4$ the parameter θ is pure imaginary, while for $q > 4$ it is real.

At $q = 4$, $z = 1$ and for the colouring polynomial $x = -\frac{1}{2}$. Thus the weights (12) are real and the last two, as well as the first, are positive. However, the third and fourth are negative. For a bipartite lattice $\mathscr{L}$ it is not difficult to show that configurations 3 and 4 occur in pairs, so can both be replaced by positive values, which proves that in this case there is a positive number of four-colourings of $\mathscr{L}$. However, since a bipartite lattice can by definition be coloured with two colours, this is not a significant result! One needs a proof that Z is positive for any lattice when θ is real.

Another intriguing point which suggests that our transformation may be relevant is the following. It is conjectured from numerical and other studies that the real zeros of the colouring polynomial of an arbitrary planar lattice tend to limits as the lattice becomes large (Kasteleyn 1975). These limits are supposed to occur at the 'Beraha numbers' $q = [2\cos(\pi/n)]^2$, $n = 2, 3, 4, \ldots$ (Tutte 1974).

From (11) we see that this corresponds to our parameter θ having the simple set of values $\theta = i\pi/n$, $n = 2, 3, 4, \ldots$.

6. The regular lattices

For the interior of regular lattices there is an obvious natural choice of $\mathscr{L}'$, namely to take the sites of $\mathscr{L}'$ to be the midpoints of the edges of $\mathscr{L}$, and to take two sites of $\mathscr{L}'$ to be adjacent if and only if the corresponding edges of $\mathscr{L}$ meet at a common site and bound a common face. All sites of $\mathscr{L}'$ are then 'internal' except at the boundaries, which is the reason for our terminology. In figures 7 and 8 we show the square and triangular lattices and their surrounding lattices (square and Kagomé, respectively).
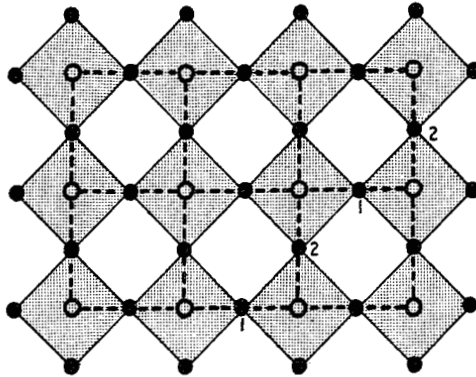
Figure 7. The square lattice $\mathscr{L}$ (open circles and broken lines) and its surrounding lattice $\mathscr{L}'$ (full circles and lines). The two classes of edges of $\mathscr{L}$, horizontal and vertical, are indicated by the numbers 1 and 2, respectively.
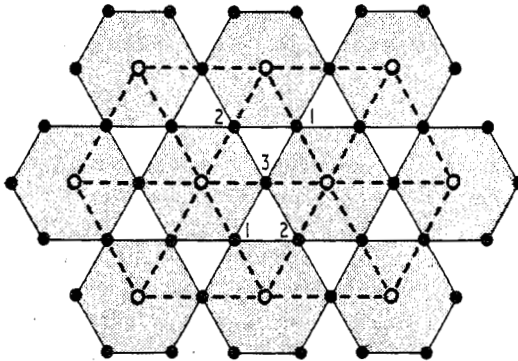
Figure 8. The triangular lattice $\mathscr{L}$ (open circles and broken lines) and its surrounding Kagomé lattice $\mathscr{L}'$ (full circles and lines).

The square lattice has two classes of edges, horizontal and vertical, which we call classes 1 and 2, respectively. The triangular lattice has three classes (1, 2 and 3), as shown in figure 8. Setting

$$s = e^{\theta/2}, \qquad t = e^{\theta/3}, \tag{16}$$

it follows from equation (12) and figure 5 that for the square lattice the vertex weights of an internal site of $\mathscr{L}'$ of class r are

$$\omega_1, \ldots, \omega_6 = 1, 1, x_r, x_r, s^{-1} + x_r s, s + x_r s^{-1}, \tag{17}$$

while for the triangular lattice they are

$$\omega_1, \ldots, \omega_6 = 1, 1, x_r, x_r, t^{-1} + x_r t^2, t + x_r t^{-2}. \tag{18}$$

The arrow configurations are still labelled $1, \ldots, 6$ as in figure 5, where the adjacent sites of $\mathscr{L}$ are drawn to the left and the right. Thus to obtain the weight of an arrow configuration at a site of $\mathscr{L}'$ on a vertical edge of the square lattice $\mathscr{L}$, it is necessary to turn figure 7 through 90° before using figure 5 and equation (17). Similar rotations are necessary for edges 1 and 2 of the triangular lattice.

For the square (triangular) lattice, external sites of $\mathscr{L}'$ have weight $s^{1/2} (t^{1/2})$ if the arrows turn to the left through the site, and weight $s^{-1/2} (t^{-1/2})$ if they turn to the right.

If we wish, we can eliminate fractional powers of e^θ by associating additional mutually inverse weights with the tips and tails of some arrows. Consider for instance the surrounding lattice $\mathscr{L}'$ of the square lattice, shown in figure 7. With every arrow pointing up and to the right associate a further weight $s^{-1}(s)$ with the site into (out of) which it points. This leaves the partition function unchanged, but on sites of type 1 multiplies ω_5, ω_6 by s, s^{-1}, respectively. On sites of type 2 it multiplies them by s^{-1}, s. We can then verify that our ice-type model for the square lattice is the same as that of Temperley and Lieb (their figure 1 and table 2, reversing vertical arrows and rotating through 135° clockwise). The only difference is that we have included the boundary conditions.

6.1. Duality

The Potts model is known to have a duality property (Potts 1952, Kihara *et al* 1954, Wu and Wang 1976). Our methods provide another way of re-deriving this.

Let $\mathscr{L}_D$ be the lattice dual to a lattice $\mathscr{L}$. Then from figures 7 and 8 it is apparent that the surrounding lattice $\mathscr{L}'$ of $\mathscr{L}$ is also the surrounding lattice of $\mathscr{L}_D$. (Here we do ignore boundary conditions.)

From (9) we can regard the Potts model partition function Z as a function of q and $x_1, x_2, x_3, \ldots$. Let us also define a Potts model on the dual lattice $\mathscr{L}_D$, with partition function Z_D and parameters $q, y_1, y_2, y_3, \ldots$ (using the obvious one-to-one correspondence between edges of $\mathscr{L}$ and $\mathscr{L}_D$). We can repeat the above reasoning to obtain the ice-type model on $\mathscr{L}'$ that is equivalent to the dual model. We find that it is again given by rules $(a), (b), (c)$, except that in (12) x_r disappears and the terms that originally did not contain a factor x_r now contain y_r.

This is equivalent to first dividing the weights (12) by x_r, then replacing x_r by y_r^{-1}. Thus we obtain the duality relation

$$Z(q; x_1, x_2, \ldots) = x_1^{e_1} x_2^{e_2} \ldots Z_D(q; x_1^{-1}, x_2^{-1}, \ldots), \tag{19}$$

where e_r is the number of edges in $\mathscr{L}$ (or $\mathscr{L}_D$) of class r.

The Potts model on the honeycomb lattice is therefore also equivalent to an ice-type model on the Kagomé lattice.

6.2. Cyclic boundary conditions

The equivalence can be extended to lattices wound on a cylinder. In this case the rule leading to (15) gives a polygon that winds around the cylinder an incorrect weight $1+1=2$. To correct this, we draw a 'seam' through the lattice $\mathscr{L}'$, avoiding all sites, from the bottom of the cylinder to the top. We then associate an extra weight $e^{\theta}(e^{-\theta})$ with arrows pointing to the left (right) on edges that cross the seam. This ensures that all polygons give the correct contribution (15).

It is not clear that the equivalence can be further generalized to toroidal boundary conditions. In one sense this does not matter, since in lattice statistics we are usually interested in the 'thermodynamic limit' of a large lattice, when boundary conditions are expected to be irrelevant. However, we feel it does clarify the equivalence to establish it exactly for finite lattices.

Note added in proof. Our 'surrounding' lattice is the same as the 'medial' graph in graph theory (Ore 1967, pp 47 and 124). The 'Whitney' polynomial is also known as the 'dichromatic' polynomial (Tutte 1967). Nagle (1968) used the method of § 2 to show the equivalence of the colouring problem with a Whitney polynomial, and in 1969 defined a staggered ice-type model similar to those that occur here.

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